Auto-parallel equation as Euler-Lagrange's equation over spaces with affine connections and metrics

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Abstract

The auto-parallel equation over spaces with affine connections and metrics [$(\overline{L}_n, g)$-spaces] is considered as a result of the application of the method of Lagrangians with covariant derivatives (MLCD) on a given Lagrangian density.

1 Introduction

In (pseudo) Riemannian spaces without torsion (V_n-spaces) the geodesic equation (identical with the auto-parallel equation $\nabla_u u = 0$) [1] can be found on the ground of the variation of an action S identified with the parameter s of a curve interpreted as its length s

$$S = \int ds + s_0 \ : \delta S = 0 \Rightarrow \nabla_u u = 0 \text{ , with } \nabla = \Gamma = \{ \ \} \ , \tag{1}$$

where $\Gamma = \{ \ \}$ is the symmetric Levi-Civita affine connection and

$$\begin{aligned} \nabla_u u &= u^i{}_{;j} \cdot u^j \cdot \partial_i \ , \quad u^i{}_{;j} = u^i{}_{,j} + \Gamma^i_{kj} \cdot u^k \ , & (2) \\ u &= u^i \cdot \partial_i = u^\alpha \cdot e_\alpha \in T(M) \ , & (3) \\ \Gamma^i_{kj} &= \{^{\ i}_{kj}\} = \frac{1}{2} \cdot g^{im} \cdot (g_{jk,m} + g_{km,j} - g_{kj,m}) \ , & (4) \\ g_{ij} &= g_{ji} \ , \ g_{ij;k} = 0 \ , \quad i, j, ... = 1, 2, ..., n \ , & (5) \\ dimM &= n \ , \quad u^i{}_{,j} = \partial u^i / \partial x^j \ , \quad u^i = dx^i / ds \ , & \end{aligned}$$

$\{\partial_i\}$ is a co-ordinate (holonomic) contravariant basis in $T(M)$, (6)
$\{e_\alpha\}$ is a non-co-ordinate (non-holonomic) contravariant basis in $T(M)$.

The same method has been used for finding out the geodesic equation in a space with contravariant and covariant affine connections (whose components

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differ not only by sign) and metrics [2] [called $(\overline{L}_n, g)$-space] [3]. Since the geodesic equation (interpreted as an equation for motion of a moving free spinless test particle in an external gravitational field) differs from the auto-parallel equation in a $(\overline{L}_n, g)$-space or in a space with an affine connection and metrics [(L_n, g)-space], the old question arises as what is the right equation for description of a free moving particle in a $(\overline{L}_n, g)$ - or a (L_n, g) -space: the geodesic equation (G) or the auto-parallel equation (A). The most authors *believe* that the geodesic equation is the more appropriate equation. One of their major arguments is that the geodesic equation is related to a variational principle (as a basic principle in classical physics) in contrast to the auto-parallel equation.

At first, the so called A-G problem has induced discussions in the case of (pseudo) Riemannian spaces with torsion (U_n-spaces, Riemann-Cartan spaces) used for describing the gravitational interaction in the Einstein-Cartan-etc. theories of gravitation and in the theory of defects in crystals [4]. The classical action (1) has been considered by Kleinert and collaborators [5] - [10], as well as by other authors [11]-[13] [14], [15] in a co-ordinate basis and in a non-co-ordinate basis in a V_n-space leading in the first basis to the geodesic equation in a V_n-space in a co-ordinate basis and in the second basis leading to the geodesic equation in a non-co-ordinate basis, where the structure coefficients $C_{\alpha\beta}{}^{\gamma}$ $\{[e_\alpha, e_\beta] = C_{\alpha\beta}{}^{\gamma} \cdot e_\gamma\}$ are related to the torsion tensor $T_{\alpha\beta}{}^{\gamma} = \Gamma_{\beta\alpha}{}^{\gamma} - \Gamma_{\alpha\beta}{}^{\gamma} - C_{\alpha\beta}{}^{\gamma}$ [16] and the equation is interpreted as an auto-parallel equation. Analogous consideration by the use of the embedding of a Riemannian space with torsion in an Euclidean space (without torsion) [17] has been made. Both the methods have been critically evaluated and the main disadvantages of the proposed variational principle have been summarized by Sá [16].

Recently, it has been proved by Iliev [18]-[24] and latter by Hartley [25] that every given affine (linear) connection in a (L_n, g)-space could vanish at a point or on a trajectory in a properly chosen co-ordinate or non-co-ordinate basis in this space. If the space has torsion, then the basis should be chosen as a non-co-ordinate basis. That means that in a (L_n, g)-space the auto-parallel equation could be written in the forms

$$\nabla_u u = u^i{}_{;j} \cdot u^j \cdot \partial_i = u^\alpha{}_{/\beta} \cdot u^\beta \cdot e_\alpha = 0 \ , \tag{7}$$

$$u^i{}_{;j} \cdot u^j = u^i{}_{,j} \cdot u^j + \Gamma^i{}_{kj} \cdot u^k \cdot u^j = 0 \ , \tag{8}$$

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{kj} \cdot \frac{dx^k}{ds} \cdot \frac{dx^j}{ds} = 0 \quad \text{for} \quad \Gamma^i_{kj} \neq 0 \ , \tag{9}$$

$$u^\alpha{}_{/\beta} \cdot u^\beta = e_\beta u^\alpha + \Gamma^\alpha_{\gamma\beta} \cdot u^\gamma \cdot u^\beta = 0 \ , \tag{10}$$

$$\frac{d^2 \overline{x}^\alpha}{ds^2} = 0 \ \text{ for } \quad \Gamma^\alpha_{\beta\gamma} = 0 \quad \text{and } \ T_{\alpha\beta}{}^{\gamma} \neq 0 \ , \tag{11}$$

where $u^\alpha = d\overline{x}^\alpha / ds = A_j{}^{\alpha} \cdot u^j$ are the components of the tangent vector $u = u^\alpha \cdot e_\alpha$ to the trajectory $x^i(\tau)$ in a non-co-ordinate (non-holonomic) contravariant basis $\{e_\alpha\} \in T(M)$ and $\overline{x}^a = A_i{}^{\alpha} \cdot dx^i$.

Therefore, in every (L_n, g)-space the auto-parallel equation (7) could be written in a special basis as an equation for motion of a moving free spinless test particle (11). "This fact leads to the conclusion that the equivalence principle in Einstein's theory of gravitation (ETG) can be considered only as a physical interpretation of a corollary of the mathematical apparatus and its validity can be extended to all spaces with affine connections and metrics. Even if a differentiable manifold has two connections (whose components differ not only by sign) for tangent and cotangent vector fields, the principle of equivalence is fulfilled for one of the two types of vector fields. Therefore, every differentiable manifold with affine connections and metrics can be used as a model for space-time in which the equivalence principle holds. On this ground, a free moving spinless test particle in a suitable basic system in a (L_n, g)- or $(\overline{L}_n, g)$-space $(n = 4)$ will fulfill an equation identical with that for the motion of a free moving spinless test particle in Newtonian mechanics or in special relativity [3]." The auto-parallel equation (interpreted in ETG, the special relativity theory, and in Newtonian mechanics as an equation for the motion of a free moving spinless test particle, and identical with the geodesic equation) can be considered as a generalization of the equation of a free moving spinless test particle in the case of (L_n, g)- or $(\overline{L}_n, g)$-spaces. The question that arises for the case of these type of spaces is the same as for the case of U_n-spaces: "Is it possible the auto-parallel equation to be derived from a variational principle?" This is an important question, inducing an important task, because there are evidences that (L_n, g)- and $(\overline{L}_n, g)$-spaces can have similar structures as the V_n-spaces for describing dynamical systems and the gravitational interaction. In such type of spaces one could use Fermi-Walker transports [26], [27] conformal transports [28], [29] and different frames of reference [30]. All these notions appear as generalizations of the corresponding notions in V_n-spaces.

On this ground, in our opinion, the failure in finding an appropriate Lagrangian formalism for obtaining the auto-parallel equation is related to the attempts of using analogous variational expressions for the action S as in the case of the geodesic equation. In $(\overline{L}_n, g)$- and (L_n, g)-spaces the auto-parallel equation has much more complicated structure (related to torsion and non-metricity) than the geodesic equation [3]. The auto-parallel equation $\nabla_u u = 0$ should be fulfilled for a given affine connection independent of a given metric. It should not depend on a metric in a manifold M (in contrast to the geodesic equation in V_n-spaces). An affine connection determines transports of other contravariant vector fields $\{\xi \in T(M)\}$ along an auto-parallel vector field u. The fact that u is an auto-parallel vector field should in some way influence the transport of a set of vector fields $\{\xi\}$ along u (if u is parallel transported, then ξ is transported along u in determined way). On the other side, an auto-parallel contravariant vector field u induces (on the ground of an existing metric in a differentiable manifold M) additional structures such as the orthogonal to it sub space $T^{\perp u}(M)$ which could also be taken into account if we wish to find the auto-parallel equation on the basis of a variational principle. In a variational principle for obtaining the auto-parallel equation $\nabla_u u = 0$ the mentioned above circumstances should be taken into account. If we use the method of Lagrangians with covariant derivatives (MLCD) we could find a solution of the G-A problem.

The method of Lagrangians with covariant derivatives is a field theoretical method, worked out as a Lagrangian formalism for tensor field theories over

(L_n, g)- and $(\overline{L}_n, g)$-spaces [31], [32]. In this sense, it is more general than the Lagrangian formalism in classical mechanics or in general relativity [33].

In this paper we consider a possible representation of the auto-parallel equation by the use of a Lagrangian formalism based on the method of Lagrangians with covariant derivatives. The method is entirely different from the methods used by Kleinert et all which are applicable in U_n-spaces but not very useful in (L_n, g)- or $(\overline{L}_n, g)$-spaces, where the nonmetricity could play as important role as the torsion. In Section 2 we recall some well known facts about the canonical representations of the parallel and the auto-parallel equations over spaces with affine connections and metrics. In Section 3 we consider degenerated Lagrangian densities and their corresponding Euler-Lagrange's equations. In Section 4 the auto-parallel equation is obtained and investigated on the basis of the MLCD as an Euler-Lagrange's equation related to a degenerated Lagrangian density with respect to a preliminary given contravariant vector field ξ. The last Section 5 comprises some concluding remarks. The most considerations are given in details (even in full details) for those readers who are not familiar with the investigated problems.

2 Canonical representation of the parallel and the auto-parallel equations

2.1 Canonical representation of a parallel equation

Let us now recall some well known facts from the differential geometry of manifolds [34]. Let a congruence $x^i(\tau, \lambda)$ be given described by the two parameters τ and λ and by the tangent vector fields ξ, $\eta \in T(M)$:

$$u := \frac{\partial}{\partial \tau} = \frac{\partial x^i}{\partial \tau} \cdot \partial_i \ , \qquad \text{and} \qquad \xi := \frac{\partial}{\partial \lambda} = \frac{\partial x^j}{\partial \lambda} \cdot \partial_j \tag{12}$$

respectively. Let us consider a parallel transport of the vector field ξ along the vector field u

$$\nabla_u \xi = f \cdot \xi \ , \qquad f \in C^r(M) \ . \tag{13}$$

An equation of this type is called recurrent equation (or recurrent relation for the vector field ξ) [35]. Three types of invariance of this equation could be found.

(a) Invariance (of type A) with respect to a change of the co-ordinates (charts) or invariance (of type B) with respect to a change of the bases in the manifold M. These types (A and B) of invariance are obvious because they follow from the index-free form of the equation.

(b) Form invariance with respect to a change of the vector ξ with a collinear to it vector $\eta := \varphi \cdot \xi$ $[\varphi = \varphi(x^k) \neq 0]$.

Proof. If we express the vector field ξ with its equivalent form $\xi = \varphi^{-1} \cdot \eta$ in the above equation, we will get

$$\nabla_u(\varphi^{-1} \cdot \eta) = u(\varphi^{-1}) \cdot \eta + \varphi^{-1} \cdot \nabla_u \eta = f \cdot \varphi^{-1} \cdot \eta \ . \tag{14}$$

After the necessary transformation of the single terms, the equation for η can be written in the form

$$\nabla_u \eta = \overline{f} \cdot \eta \ , \qquad \overline{f} := f - u(\log \varphi^{-1}) \ . \tag{15}$$

Therefore, the parallel equation for ξ does not change its form by the change of ξ with a collinear vector field η.

(c) Form invariance with respect to a change of the parameter λ determining ξ.

Proof. The change of the parameter λ with a new parameter $\sigma = \sigma(\lambda)$ with $\lambda = \lambda(\sigma)$, and $\frac{d\sigma}{d\tau} = 0$, leads to the relations

$$\xi^i = \frac{\partial x^i}{\partial \lambda} = \frac{\partial x^i}{\partial \sigma} \cdot \frac{d\sigma}{d\lambda} = \overline{\xi}^{\,i} \cdot \frac{d\sigma}{d\lambda} \ , \qquad \overline{\xi}^{\,i} = \frac{\partial x^i}{\partial \sigma} \ , \qquad \frac{d\sigma}{d\lambda} :\neq 0 \ . \tag{16}$$

$$\xi^i{}_{,j} = \left(\frac{\partial x^i}{\partial \lambda}\right)_{,j} = \left(\frac{\partial x^i}{\partial \sigma} \cdot \frac{d\sigma}{d\lambda}\right)_{,j} = \left(\frac{\partial x^i}{\partial \sigma}\right)_{,j} \cdot \frac{d\sigma}{d\lambda} + \frac{\partial x^i}{\partial \sigma} \cdot \left(\frac{d\sigma}{d\lambda}\right)_{,j} \ , \tag{17}$$

$$\xi^i{}_{,j} \cdot u^j = \xi^i{}_{,j} \cdot \frac{\partial x^j}{\partial \tau} = \frac{\partial \overline{\xi}^{\,i}}{\partial \tau} \frac{d\sigma}{d\lambda} + \overline{\xi}^{\,i} \cdot \frac{\partial}{\partial \tau} \left(\frac{d\sigma}{d\lambda}\right) \ . \tag{18}$$

Since σ and λ are independent of the parameter τ the second term at the right of the equations vanishes. Therefore,

$$\xi^i{}_{,j} \cdot u^j = \xi^i{}_{,j} \cdot \frac{\partial x^j}{\partial \tau} = \frac{\partial \overline{\xi}^{\,i}}{\partial \tau} \frac{d\sigma}{d\lambda} = \frac{\partial \xi^i}{\partial \tau} \tag{19}$$

and the parallel equation for ξ will be expressed in terms of $\overline{\xi}$ in the form

$$\xi^i{}_{;j} \cdot u^j = \xi^i{}_{,j} \cdot u^j + \Gamma^i_{jk} \cdot \xi^j \cdot u^k = (\frac{\partial \overline{\xi}^{\,i}}{\partial \tau} + \Gamma^i_{jk} \cdot \overline{\xi}^{\,j} \cdot u^k) \cdot \frac{d\sigma}{d\lambda} = f \cdot \overline{\xi}^{\,i} \cdot \frac{d\sigma}{d\lambda} \ . \tag{20}$$

Therefore, we have

$$\frac{\partial \overline{\xi}^{\,i}}{\partial \tau} + \Gamma^i_{jk} \cdot \overline{\xi}^{\,j} \cdot u^k = \overline{\xi}^{\,i}{}_{;j} \cdot u^j = f \cdot \overline{\xi}^{\,i} \qquad \cong \qquad \nabla_u \overline{\xi} = f \cdot \overline{\xi} \ , \quad \overline{\xi} = \overline{\xi}^{\,i} \cdot \partial_i \ . \tag{21}$$

2.2 Canonical representation of an auto-parallel equation

Let us now consider the auto-parallel equation $\nabla_u u = f \cdot u$ as a special case of a parallel equation for $\xi = u$, $f = f(x^k(\tau))$. In this case

$$\lambda = \tau \quad , \quad u = \frac{d}{d\tau} \quad , \quad u^i = \frac{dx^i}{d\tau} \quad \text{and} \quad \sigma = \sigma(\tau) \ , \quad \tau = \tau(\sigma) \ , \ \frac{d\sigma}{d\tau} \neq 0 \ . \tag{22}$$

Then

$$u^i = \frac{dx^i}{d\sigma} \cdot \frac{d\sigma}{d\tau} = \overline{u}^i \cdot \frac{d\sigma}{d\tau} \ , \quad \overline{u}^i = \frac{dx^i}{d\sigma} \ , \tag{23}$$

$$u^i{}_{,j} \cdot u^j = \frac{d\overline{u}^i}{d\sigma} \cdot \left(\frac{d\sigma}{d\tau}\right)^2 + \overline{u}^i \cdot \frac{d^2\sigma}{d\tau^2} \ , \tag{24}$$

$$u^i{}_{;j} \cdot u^j - f \cdot u^i = \left(\frac{d\sigma}{d\tau}\right)^2 \cdot \left(\frac{d\overline{u}^i}{d\sigma} + \Gamma^i_{jk} \cdot \overline{u}^j \cdot \overline{u}^k\right) + \overline{u}^i \cdot \left(\frac{d^2\sigma}{d\tau^2} - f(\tau) \cdot \frac{d\sigma}{d\tau}\right) = 0 \; . \tag{25}$$

One can chose as condition for determining the function $\sigma = \sigma(\tau)$ as a function of τ the vanishing of the last term of the above equation

$$\frac{d^2\sigma}{d\tau^2} - f(\tau) \cdot \frac{d\sigma}{d\tau} = 0 \; . \tag{26}$$

The last equation is of the type

$$y' - f \cdot y = 0 \; , \;\; \text{where} \qquad y = \frac{d\sigma}{d\tau} \; , \qquad y' = \frac{d^2\sigma.}{d\tau^2} \; . \tag{27}$$

Then

$$\sigma = \sigma_0 + \sigma_1 \cdot \int \exp\left(\int f(\tau) \cdot d\tau\right) \cdot d\tau \; , \quad \sigma_0 = \text{ const.}, \quad \sigma_1 = \text{ const.} \tag{28}$$

After the introduction of the new parameter $\sigma = \sigma(\tau)$ (called canonical parameter), the auto-parallel equation will have the form

$$\nabla_{\overline{u}}\overline{u} = 0 \quad \cong \quad \overline{u}^i{}_{;j} \cdot \overline{u}^j = 0 \quad , \quad \overline{u} = \frac{d}{d\sigma} \; , \quad \overline{u}^i = \frac{dx^i}{d\sigma} \; . \tag{29}$$

The last equation for $\overline{u}$ is called auto-parallel equation in canonical form.

3 Degenerated Lagrangian densities and Euler-Lagrange's equations

A degenerated Lagrangian density with respect to field variables $V^A{}_B$ is a Lagrangian density $\mathbf{L}$ of the type

$$\mathbf{L} = \sqrt{-d_g} \cdot L(K^A{}_B, \, K^A{}_{B;i}, \, K^A{}_{B;i;j}, \, V^C{}_D) \; , \tag{30}$$

where

$$V^C{}_D \neq K^A{}_B \; . \tag{31}$$

The functions $K^A{}_B(x^k)$ and $V^C{}_D(x^k)$ are components of tensor fields with finite rank. The symbol $_{;k}$ is denoted the covariant derivative, defined by the use of given affine connections, with respect to a co-ordinate x^k or to a basis e_k (∂_k) . Therefore, a degenerated Lagrangian invariant L is an invariant depending only on the field variables $V^C{}_D$ and not depending on their first and second (partial or covariant) derivatives, i.e.

$$L = L(K^A{}_B, \, K^A{}_{B;i}, \, K^A{}_{B;i;j}, \, V^C{}_D) \; . \tag{32}$$

If we consider the field variables $V^C{}_D$ as dynamic field variables, then the functional variation of the Lagrangian density $\mathbf{L}$, leading to the corresponding Euler-Lagrange's equations, has the form [32]

$$\frac{\delta \mathbf{L}}{\delta V^C{}_D} = \sqrt{-d_g} \cdot \frac{\partial L}{\partial V^C{}_D} \qquad \text{for} \quad V^C{}_D \neq g_{ij} \; (\neq K^A{}_B), \tag{33}$$

$$\frac{\delta \mathbf{L}}{\delta g_{kl}} = \sqrt{-d_g} \cdot (\frac{\partial L}{\partial g_{kl}} + \frac{1}{2} \cdot L \cdot g^{\overline{kl}}) \quad \text{for} \quad V^C{}_D = g_{kl} \quad (\neq K^A{}_B) \ . \tag{34}$$

The Euler-Lagrange equations (if they could exist) for $V^C{}_D$ could be found in the form

(a) For $V^C{}_D \neq g_{kl}$:

$$\frac{\partial L}{\partial V^C{}_D} = 0 \ . \tag{35}$$

(b) For $V^C{}_D = g_{kl}$:

$$\frac{\partial L}{\partial g_{kl}} + \frac{1}{2} \cdot L \cdot g^{\overline{kl}} = 0 \ . \tag{36}$$

Let us now consider both the cases separately to each other.

3.1 Degenerated Lagrangian densities with respect to non-metric field variables

Let $V^C{}_D \neq g_{kl}$ be given fulfilling the Euler-Lagrange equations

$$\frac{\partial L}{\partial V^C{}_D} = G_C{}^D = 0 \ . \tag{37}$$

One can distinguish sub cases following as solutions of the equations (35):

(a) L is independent of the field variables $V^C{}_D$. This conclusion from the equations contradicts to the prerequisite for the structure of L.

(b) L is linearly dependent on $V^C{}_D$. Then L could be written in the form

$$\begin{aligned} L &= L_0 + F(K^A{}_B, K^A{}_{B;i}, K^A{}_{B;i;j}) \cdot G_C{}^D (K^A{}_B, K^A{}_{B;i}, K^A{}_{B;i;j}) \cdot V^C{}_D , \\ L_0 &= \text{const.} \end{aligned} \tag{38}$$

The Euler-Lagrange equations for $V^C{}_D$ degenerate to conditions for the other field variables $K^A{}_B$

$$\frac{\partial L}{\partial V^C{}_D} = F \cdot G_C{}^D = 0 \ . \tag{39}$$

The last equations, among with the Euler-Lagrange equations for the field variables $K^A{}_B$, form a system of differential equations for all field variables in L considered as dynamic field variables. Here $V^C{}_D$ take the role of Lagrangian multipliers. If $K^A{}_B$ are not considered as dynamic field variables, i.e. if they are assumed as preliminary given non-dynamic field variables, then the equations (39) appear as additional conditions (constrains) for $K^A{}_B$. If (39) are fulfilled, then $L = L_0 =$ const. Furthermore, if we interpret L as the pressure $p = L$ in a physical system [36], [37] then the existence of Euler-Lagrange equations for $V^C{}_D$ leads to establishing of a constant or vanishing pressure [$L = L_0 = p_0 =$ const. $(\neq 0, = 0)$] in the system. Therefore, if we wish to consider a system with $p = p_0 =$ const.$(\neq 0, = 0)$, we can introduce a Lagrangian invariant L of type (b) and then we can find the Euler-Lagrange equations for the dynamic

field variables $V^C{}_D$ and their corresponding energy-momentum tensors by the use of the method of Lagrangians with covariant derivatives (MLCD).

(c) L does not depend linearly on $V^C{}_D$. Then L could have the general form

$$L = L(K^A{}_B,\, K^A{}_{B;i},\, K^A{}_{B;i;j},\, V^C{}_D) \ . \tag{40}$$

The Euler-Lagrange equations for $V^C{}_D$ degenerate in this case to algebraic equations for $V^C{}_D$

$$G_C{}^D = \frac{\partial L}{\partial V^C{}_D} = 0 \ . \tag{41}$$

3.2 Degenerated Lagrangian densities with respect to metric field variables

Let $V^C{}_D = g_{kl}$ be given fulfilling the Euler-Lagrange equations

$$\frac{\partial L}{\partial g_{kl}} + \frac{1}{2} \cdot L \cdot g^{\overline{kl}} = 0 \ . \tag{42}$$

The invariant L has to obey the condition

$$\frac{\partial L}{\partial g_{ij}} \cdot g_{ij} + \frac{n}{2} \cdot L = 0 \ , \tag{43}$$

i.e. it should be a homogeneous function of degree $h = -\frac{n}{2}$ with respect to the metric field components g_{ij}

$$\frac{\partial L}{\partial g_{ij}} \cdot g_{ij} = -\frac{n}{2} \cdot L \ . \tag{44}$$

We can distinguish some sub cases induced by the possible solutions of the Euler-Lagrange equations.

(a) L is linearly dependent on g_{ij}, i.e. L could be represented in the form

$$L = L_0 + k \cdot F \cdot G^{\overline{ij}} \cdot g_{ij} \ , \tag{45}$$

where

$$L_0 = \text{ const.}, \qquad k = \text{ const.}, \qquad G^{\overline{ij}} = G^{\overline{ji}} \ . \tag{46}$$

From (45), it follows that

$$\frac{\partial L}{\partial g_{ij}} = k \cdot F \cdot G^{\overline{ij}} \tag{47}$$

and

$$\frac{\partial L}{\partial g_{ij}} \cdot g_{ij} = k \cdot F \cdot G^{\overline{ij}} \cdot g_{ij} = L - L_0 = -\frac{n}{2} \cdot L : \qquad (1 + \frac{n}{2}) \cdot L = L_0 \ . \tag{48}$$

If $L_0 = 0$, then either $n = -2$ (which contradicts to the condition $dimM = n \geq 1$), or $L = 0$. Then,

$$\frac{\partial L}{\partial g_{ij}} \cdot g_{ij} = 0 : \qquad\qquad k \cdot F \cdot G^{\overline{ij}} = 0 \ . \tag{49}$$

If $L_0 \neq 0$, then

$$L = \frac{2}{n+2} \cdot L_0 = \text{ const.}, \qquad k \cdot F \cdot G^{\overline{ij}} \quad + \frac{1}{n+2} \cdot L_0 \cdot g^{\overline{ij}} = 0 \ , \tag{50}$$

$$g^{ij} = - \ (n+2) \cdot \frac{k}{L_0} \cdot F \cdot G^{ij} \ . \tag{51}$$

The explicit form of g_{ij} is determined by the functions F and G^{ij} as functions of the field variables $K^A{}_B$ and their first and second covariant derivatives.

(b) L does not dependent linearly on g_{ij}. At the same time, L should fulfil the homogeneous conditions (44). The Euler-Lagrange equations for g_{ij} appear as algebraic equations for g_{ij}. Together with the Euler-Lagrange equations for the field variables $K^A{}_B$ they determine a system of differential equations for the field variables $(K^A{}_B, \, g_{ij})$.

4 Auto-parallel equation as Euler-Lagrange's equation

4.1 Euler-Lagrange's equation and auto-parallel equation for a vector field u

Every Lagrangian invariant L could be interpreted as the pressure p of the considered physical system in a 4-dimensional space with affine connections and metrics used as a model of space-time [31], [36].

Let us define a Lagrangian invariant in the form

$$L := p = p_0 + a_0 \cdot \rho \cdot e + h_0 \cdot g(\nabla_u \rho u, \xi) \ , \tag{52}$$

where

$$\begin{aligned} p_0 &= \ \text{const.}, \quad a_0 = \text{ const.}, \quad h_0 = \text{ const.}, & (53)\\ \rho &= \ \rho(x^k) \in C^r \ , \quad r \geq 2 \ , \qquad u, \, \xi \in T(M) \ , & (54)\\ e &= \ g(u,u) \neq 0 \ , \qquad g = g_{ij} \cdot dx^i . dx^j \ . & (55) \end{aligned}$$

The pressure p could also be written in the form

$$p = p_0 + f \cdot \rho + b \cdot (u\rho) \ , \tag{56}$$

where

$$\begin{aligned} f &= \ a_0 \cdot e + h_0 \cdot g(a, \xi) \ , \qquad a = \nabla_u u \quad , & (57)\\ b &= \ h_0 \cdot l \ , \qquad l = g(u, \xi) \ . & (58) \end{aligned}$$

The constant p_0 is a constant pressure, $\rho \cdot e$ is interpreted as the kinetic energy density, u is the velocity of the points in the system, ξ is a contravariant vector transported along u.

In a co-ordinate basis f and b will have the forms respectively

$$\begin{aligned} f &= \ g_{\overline{kl}} \cdot (a_0 \cdot u^k \cdot u^l + h_0 \cdot u^k{}_{;m} . u^m \cdot \xi^l) \ , & (59)\\ b &= \ h_0 \cdot g_{\overline{kl}} \cdot u^k \cdot \xi^l \ . & (60) \end{aligned}$$

By the use of the method of Lagrangians with covariant derivatives we can find the Euler-Lagrange equations for the field variables ρ, u, ξ, and g as well as the corresponding energy-momentum tensors. Moreover, we can consider every sub set of the set $\{\rho,\ u,\ \xi,\ g\}$ as dynamic field variables (for which the Euler-Lagrange's equations should be found), where the rest of the field variables are considered as non-dynamic field variables. In the further investigations we will consider all field variables $\{\rho,\ u,\ \xi,\ g\}$ as dynamic field variables. At that, the Lagrangian invariant L appears as a degenerated Lagrangian invariant with respect to the metric tensor components g_{ij} and the components ξ^i of the contravariant vector ξ.

For finding out the Euler-Lagrange equations for ρ, u, ξ, and g, we have to find the explicit form of some auxiliary expressions. Let us make a list of them.

$$\frac{\partial e}{\partial u^i} = 2 \cdot g_{\overline{i}\overline{k}} \cdot u^k = 2 \cdot u_{\overline{i}} \ , \qquad \frac{\partial e}{\partial g_{ij}} = u^{\overline{i}} \cdot u^{\overline{j}} \ , \tag{61}$$

$$\frac{\partial g(a,\xi)}{\partial u^i} = g_{\overline{k}\overline{l}} \cdot \xi^l \cdot u^k{}_{;i} \ , \qquad \frac{\partial g(a,\xi)}{\partial u^i{}_{;j}} = g_{\overline{i}\overline{k}} \cdot \xi^k \cdot u^j \ , \qquad \frac{\partial g(a,\xi)}{\partial \xi^i} = g_{\overline{i}\overline{k}} \cdot .a^k \ , \tag{62}$$

$$\frac{\partial g(a,\xi)}{\partial g_{ij}} = \frac{1}{2} \cdot (a^{\overline{i}} \cdot \xi^{\overline{j}} + a^{\overline{j}} \cdot \xi^{\overline{i}}) \ , \tag{63}$$

$$\frac{\partial b}{\partial \rho} = 0 \ , \quad \frac{\partial f}{\partial \rho} = 0 \ , \quad \frac{\partial b}{\partial \rho_{,i}} = \frac{\partial b}{\partial (\rho_{,i})} = 0 \ , \tag{64}$$

$$\frac{\partial b}{\partial u^i} = h_0 \cdot g_{\overline{i}\overline{k}} \cdot \xi^k \ , \quad \frac{\partial b}{\partial \xi^i} = h_0 \cdot g_{\overline{i}\overline{k}} \cdot u^k \ , \quad \frac{\partial b}{\partial g_{ij}} = \frac{1}{2} \cdot h_0 \cdot (u^{\overline{i}} \cdot \xi^{\overline{j}} + u^{\overline{j}} \cdot \xi^{\overline{i}}) \ , \tag{65}$$

$$\frac{\partial (u\rho)}{\partial u^i} = b \cdot \rho_{,i} \ , \qquad \delta u = u^i{}_{;i} \quad . \tag{66}$$

4.1.1 Euler-Lagrange's equation for the scalar function ρ

The Euler-Lagrange equation for ρ has the form [31]

$$\frac{\partial p}{\partial \rho} - (\frac{\partial p}{\partial \rho_{,i}})_{;i} + q_i \cdot \frac{\partial \rho}{\partial \rho_{,i}} = 0 \ , \tag{67}$$

where

$$\frac{\partial p}{\partial \rho} = f \ , \qquad \frac{\partial p}{\partial \rho_{,i}} = b \cdot u^i \ , \qquad (\frac{\partial p}{\partial \rho_{,i}})_{;i} = ub + b \cdot \delta u = b_{,i} \cdot u^i + b \cdot u^i{}_{;i} \quad , \tag{68}$$

$$q_i = T_{ki}{}^i - \frac{1}{2} \cdot g^{\overline{k}\overline{l}} \cdot g_{kl;i} + g^l_k \cdot g^k_{l;i} \ , \qquad q = q_j \cdot u^j \ , \quad q_i \cdot \frac{\partial \rho}{\partial \rho_{,i}} = b \cdot q \quad , \tag{69}$$

After substituting the explicit forms of the single terms in the Euler-Lagrange's equation, we obtain the explicit form of the Euler-Lagrange's equation for the scalar function ρ in the form

$$ub = f + (q - \delta u) \cdot b \quad , \tag{70}$$

or in the forms

$$h_0 \cdot ul = a_0 \cdot e + h_0 \cdot g(a, \xi) + h_0 \cdot (q - \delta u) \cdot l \ , \tag{71}$$

$$l_{,i} \cdot u^i = g_{\overline{kl}} \cdot \left\{ \frac{a_0}{h_0} \cdot u^k \cdot u^l + [u^k{}_{;m} \cdot u^m \cdot \xi^l + (q_i \cdot u^i - u^i{}_{;i}) \cdot u^k \cdot \xi^l] \right\} \ . \tag{72}$$

It follows from the explicit form of the Euler-Lagrange equation for ρ that this equation does not depend on ρ and could be considered as a condition for $l = g(u, \xi)$.

4.1.2 Euler-Lagrange's equations for the contravariant vector field u

The Euler-Lagrange equations for the vector field u have the form

$$\frac{\partial p}{\partial u^i} - (\frac{\partial p}{\partial u^i{}_{;j}})_{;j} + q_j \cdot \frac{\partial p}{\partial u^i{}_{;j}} = 0 \quad , \tag{73}$$

where

$$\frac{\partial p}{\partial u^i} = h_0 \cdot l \cdot \rho_{,i} + g_{\overline{ik}} \cdot (2 \cdot a_0 \cdot \rho \cdot u^k + h_0 \cdot \rho_{,j} \cdot u^j \cdot \xi^k) + h_0 \cdot \rho \cdot g_{\overline{kl}} \cdot u^k{}_{;i} \cdot \xi^l \ , \tag{74}$$

$$\frac{\partial p}{\partial u^i{}_{;j}} = \rho \cdot \frac{\partial f}{\partial u^i{}_{;j}} = h_0 \cdot \rho \cdot g_{\overline{ik}} \cdot \xi^k \cdot u^j \quad , \tag{75}$$

$$q_j \cdot \frac{\partial p}{\partial u^i{}_{;j}} = h_0 \cdot \rho \cdot q \cdot g_{\overline{ik}} \cdot \xi^k \quad , \tag{76}$$

$$\begin{aligned} (\frac{\partial p}{\partial u^i{}_{;j}})_{;j} \quad &= \quad h_0 \cdot \{ g_{\overline{ik}} \cdot [\rho_{,j} \cdot u^j \cdot \xi^k + \rho \cdot (\xi^k{}_{;j} \cdot u^j + u^m{}_{;m} \cdot \xi^k - g^k_{l;j} \cdot u^j \cdot \xi^l)] + \\ &\quad + \rho \cdot (g_{\overline{ik}})_{;j} \cdot u^j \cdot \xi^k \} \ . \end{aligned} \tag{77}$$

After substituting the above expressions in (73), we obtain the explicit form of the Euler-Lagrange equations for u^i as equations for the components ξ^i of the contravariant vector field ξ

$$\begin{aligned} \xi^i{}_{;j} \cdot u^j \quad &= \quad l \cdot (log\, \rho)_{,j} \cdot g^{ji} + 2 \cdot \frac{a_0}{h_0} \cdot u^i + (q - \delta u) \cdot \xi^i + g_{\overline{kl}} \cdot u^k{}_{;j} \cdot g^{ji} \cdot \xi^l - \\ &\quad - g^{ij} \cdot (g_{\overline{jk}})_{;m} \cdot u^m \cdot \xi^k + g^i_{k;j} \cdot u^j \cdot \xi^k \ . \end{aligned} \tag{78}$$

The last equations determine the transport of the contravariant vector field ξ along the vector field u.

4.1.3 Euler-Lagrange's equations for the contravariant vector field ξ

Since p depends only on the components ξ^i of the vector field ξ and does not depend on its covariant derivatives, p appears as a degenerated Lagrangian invariant with respect to the field variables ξ^i. The Euler-Lagrange equations for the vector field ξ have the form

$$\frac{\partial p}{\partial \xi^i} = 0 \ , \tag{79}$$

where

$$\frac{\partial p}{\partial \xi^i} = \rho \cdot \frac{\partial f}{\partial \xi^i} + (u\rho) \cdot \frac{\partial b}{\partial \xi^i} \ , \qquad u\rho = \rho_{,j} \cdot u^j \ , \tag{80}$$

$$\frac{\partial b}{\partial \xi^i} = h_0 \cdot g_{\overline{ik}} \cdot u^k \ , \qquad \frac{\partial f}{\partial \xi^i} = h_0 \cdot g_{\overline{ik}} \cdot a^k \ , \tag{81}$$

$$\frac{\partial p}{\partial \xi^i} = h_0 \cdot g_{\overline{ik}} \cdot [\rho \cdot a^k + (u\rho) \cdot u^k] \ . \tag{82}$$

The explicit form of the Euler-Lagrange equations for ξ^i follows in the form of equations for the components u^i of the contravariant vector field u

$$a^i = u^i{}_{;j} \cdot u^j = -[(log\, \rho)_{,j} \cdot u^j] \cdot u^i \ , \tag{83}$$

or in index-free form

$$a = \nabla_u u = f \cdot u \ , \qquad f = -[(log\, \rho)_{,j} \cdot u^j] \ . \tag{84}$$

The last equation is exactly the auto-parallel equation for the vector field u in a non-canonical form. After changing the parameter τ of the curve $x^k(\tau)$, to which $u = \frac{d}{d\tau}$ is a tangent vector, the equation could be written in its canonical form $\nabla_{\overline{u}}\overline{u} = 0$, where

$$\overline{u} = \frac{d}{d\sigma} \ , \qquad \sigma = \sigma_1 \cdot \int \exp\left(\int f(\tau) \cdot d\tau\right) \cdot d\tau + \sigma_0 \ , \qquad \sigma_1 = \text{ const.}, \sigma_0 = \text{ const.}, \tag{85}$$

$$\sigma = \rho_0 \cdot \int \frac{1}{\rho} \cdot d\tau + \sigma_0 \ , \qquad \rho_0 = \text{ const.} \tag{86}$$

4.1.4 Euler-Lagrange's equations for the metric tensor field g

The pressure p is also a degenerated Lagrangian invariant with respect to the field variables g_{ij}. The corresponding Euler-Lagrange equations for g_{ij} could be written in the form

$$\frac{\partial p}{\partial g_{ij}} + \frac{1}{2} \cdot p \cdot g^{\overline{ij}} = 0 \ , \tag{87}$$

where

$$\frac{\partial p}{\partial g_{ij}} = \rho \cdot \frac{\partial f}{\partial g_{ij}} + (u\rho) \cdot \frac{\partial b}{\partial g_{ij}} \ , \tag{88}$$

$$\frac{\partial f}{\partial g_{ij}} = a_0 \cdot u^{\overline{i}} \cdot u^{\overline{j}} + h_0 \cdot (a^{\overline{i}}.\xi^{\overline{j}} + a^{\overline{j}}.\xi^{\overline{i}}) \ , \qquad \frac{\partial b}{\partial g_{ij}} = \frac{1}{2} \cdot h_0 \cdot (u^{\overline{i}} \cdot \xi^{\overline{j}} + u^{\overline{j}} \cdot \xi^{\overline{i}}) \ . \tag{89}$$

The explicit form of the Euler-Lagrange equations for g_{ij} follows in the form

$$\begin{aligned} g^{ij} &= -\frac{2}{p} \cdot [a_0 \cdot \rho \cdot u^i \cdot u^j + \frac{1}{2} \cdot h_0 \cdot \rho \cdot (a^i \cdot \xi^j + a^j \cdot \xi^i) + \\ &\quad + \frac{1}{2} \cdot h_0 \cdot (\rho_{,m} \cdot u^m) \cdot (u^i \cdot \xi^j + u^j \cdot \xi^i)] \ . \end{aligned} \tag{90}$$

Now, we can write down in a table the Euler-Lagrange equations for the field variables ρ, u, ξ, and g

Field variable	Euler-Lagrange's equations
ρ	$l_{,i} \cdot u^i = g_{\overline{kl}} \cdot [\frac{a_0}{h_0} \cdot u^k \cdot u^l + a^k \cdot \xi^l + (q_m \cdot u^m - u^m{}_{;m}) \cdot u^k \cdot \xi^l]$
u	$\xi^i{}_{;j} \cdot u^j = l \cdot (log\, \rho)_{,j} \cdot g^{ji} + 2 \cdot \frac{a_0}{h_0} \cdot u^i + (q - \delta u) \cdot \xi^i + g_{\overline{kl}} \cdot u^k{}_{;j} \cdot g^{ji} \cdot \xi^l -$ $- g^{ij} \cdot (g_{\overline{jk}})_{;m} \cdot u^m \cdot \xi^k + g^i_{k;j} \cdot u^j \cdot \xi^k$
ξ	$a^i = u^i{}_{;j} \cdot u^j = -[(log\, \rho)_{,j} \cdot u^j] \cdot u^i = -\frac{1}{\rho} \cdot (\rho_{,m} \cdot u^m) \cdot u^i$
g	$g^{ij} = -\frac{2}{p} \cdot [a_0 \cdot \rho \cdot u^i \cdot u^j + \frac{1}{2} \cdot h_0 \cdot \rho \cdot (a^i \cdot \xi^j + a^j \cdot \xi^i) +$ $+ \frac{1}{2} \cdot h_0 \cdot (\rho_{,m} \cdot u^m) \cdot (u^i \cdot \xi^j + u^j \cdot \xi^i)]$

4.2 Consequences from the Euler-Lagrange equations

Let us now consider the system of the Euler-Lagrange equations for ρ, u, ξ, and g.

From (83) and (90), it follows that

$$g^{ij} = -\frac{2 \cdot \rho}{p} \cdot a_0 \cdot u^i \cdot u^j \ . \tag{91}$$

After contraction of g^{ij} with $u_{\overline{i}} \cdot u_{\overline{j}}$, we have

$$e = -\frac{2 \cdot \rho}{p} \cdot a_0 \cdot e^2 : \quad p = -2 \cdot \rho \cdot a_0 \cdot e \ . \tag{92}$$

After contraction of g^{ij} with $g_{\overline{ij}}$, it follows

$$g^{ij} \cdot g_{\overline{ij}} = n = -\frac{2 \cdot \rho}{p} \cdot a_0 \cdot e : \quad n \cdot p = -2 \cdot \rho \cdot a_0 \cdot e \ . \tag{93}$$

From the last two expressions for p, we obtain

$$(n-1) \cdot p = 0 \ . \tag{94}$$

Therefore, if the Euler-Lagrange equations for g_{ij} have to be fulfilled, then either the dimension $dim M$ of the manifold M should be equal to 1 ($dim\, M = 1$) or the pressure p should vanish. The first case ($n = 1$) leads to consideration of an auto-parallel curve as one-dimensional manifold and a tangent to it vector u. In the second case ($p = 0$), we obtain the relation $\rho \cdot a_0 \cdot e = 0$ with $\rho \neq 0$, $e \neq 0$. Then only a_0 should vanish ($a_0 = 0$), and g^{ij} (respectively g_{ij}) could not be determined uniquely.

Therefore, if we wish to investigate a Lagrangian system with the given Lagrangian invariant (52) in a manifold M with $dim\, M > 1$, with $p \neq 0$, and determined metric tensor g, we should consider g as a given non-dynamic field variable or we can choose one of the following possibilities:

(a) We should consider some of the other field variables (u, ξ, ρ) as non-dynamic (fixed, given) field variables.

(b) We should add additional terms to p leading to well determined Euler-Lagrange equations for g (as it is the case in the Einstein theory of gravitation, where $p_E := p + c_0 \cdot R$, $c_0 =$ const., $R = g^{ij} \cdot R_{ij} = g^{ij} \cdot g^k_l \cdot R^l{}_{ijk}$).

At the same time, from (72) and (83), the condition for $l = g(u, \xi)$ in the case $a_0 = 0$ follows in the form

$$ul + h_0 \cdot [u(log\, \rho) + (\delta u - q)] \cdot l = \frac{a_0}{h_0} \cdot e = 0 \ , \tag{95}$$

allowing the trivial solution $l = 0$, i.e. u and ξ could be orthogonal to each other if $p = 0$.

4.2.1 Euler-Lagrange's equations for ρ, u, and ξ

If we consider only the field variables ρ, u, and ξ as dynamic field variables, then the corresponding Euler-Lagrange equations (ELEs) will have the forms

Field variable	Euler-Lagrange's equations
ρ	$ul + [u(log\, \rho) + (\delta u - q)] \cdot l = \frac{a_0}{h_0} \cdot e$
u	$\xi^i{}_{;j} \cdot u^j = l \cdot (log\, \rho)_{,j} \cdot g^{ji} + 2 \cdot \frac{a_0}{h_0} \cdot u^i + (q - \delta u) \cdot \xi^i + g_{\overline{kl}} \cdot u^k{}_{;j} \cdot g^{ji} \cdot \xi^l -$ $- g^{ij} \cdot (g_{\overline{jk}})_{;m} \cdot u^m \cdot \xi^k + g^i_{k;j} \cdot u^j \cdot \xi^k$
ξ	$a^i = u^i{}_{;j} \cdot u^j = -[(log\, \rho)_{,j} \cdot u^j] \cdot u^i = -\frac{1}{\rho} \cdot (\rho_{,m} \cdot u^m) \cdot u^i$

Since the Euler-Lagrange equations are valid for every preliminary given affine connections and metrics, we can search for their solutions in different type of spaces with affine connections and metrics. If u is a tangent vector $u = \frac{d}{d\tau}$ of a curve with parameter τ, then the Euler-Lagrange equation for ρ appears as an ordinary differential equation of first order for the scalar product $l = g(u, \xi)$ of u and ξ

$$\frac{dl}{d\tau} + g(\tau) \cdot l = \overline{k}_0 \cdot e \ , \tag{96}$$

where

$$g(\tau) = [u(log\,\rho) + (\delta u - q)] \ , \qquad \overline{k}_0 = \frac{a_0}{h_0} = \text{const.} \neq 0 \ . \tag{97}$$

The exact solution of the equation (96) for $l(\tau)$ is :

$$l\,(\tau) = exp(-\frac{1}{2} \cdot g \cdot \tau^2) \cdot [\overline{k}_0 \cdot e \cdot \int exp(\frac{1}{2} \cdot g \cdot \tau^2) \ \cdot d\tau + c_1] \ , \qquad c_1 = \ \text{const.} \tag{98}$$

Special case: $a_0 := 0 : p = p_0 + h_0 \cdot g(\nabla_u \rho u, \xi) = p_0 + h_0 \cdot [(u\rho) \cdot l + \rho \cdot g(a, \xi)]$.
The Euler-Lagrange equations for ρ, u, and ξ will have the form

$$ul + [u(log\,\rho) + (\delta u - q)] \cdot l = 0 \ , \tag{99}$$

$$a = \nabla_u u = -[u(log\,\rho)] \cdot u \ \ , \tag{100}$$

$$\nabla_u \xi = l \cdot \overline{g}(log\,\rho) + (q - \delta u) \cdot \xi + (\xi)(g)(k_0) - \overline{N} \tag{101}$$

where

$$k_0 = u^i{}_{;l} \cdot g^{lj} \cdot \partial_i \otimes \partial_j \qquad , \qquad \overline{N} = [g^{ij} \cdot (g_{\overline{jk}})_{;m} \cdot u^m - g^i_{k;j} \cdot u^j] \cdot \xi^k \cdot \partial_i \ . \tag{102}$$

If the Euler-Lagrange equations (99) $\div$ (101) are fulfilled, then $p = p_0 =$ const. The auto-parallel equation for u appears in a non-canonical form. The equation (99) has as trivial solution $l = 0$, i.e. ξ could be chosen as an orthogonal to u contravariant vector field.

Special case: $\rho = \rho_0 :=$ const. $\neq 0 : p = p_0 + a_0 \cdot \rho_0 \cdot e + h_0 \cdot \rho_0 \cdot g(a, \xi)$.
The Euler-Lagrange equations for u and ξ will have the form

$$\nabla_u u = a = 0 \ , \tag{103}$$

$$\nabla_u \xi = 2 \cdot \frac{a_0}{h_0} \cdot u + (q - \delta u) \cdot \xi + (\xi)(g)(k_0) - \overline{N} \ \ . \tag{104}$$

If (103) and (104) are fulfilled, then $p = p_0 + a_0 \cdot \rho_0 \cdot e$. The auto-parallel equation appears in its canonical form. If u is a normalized vector field [$g(u, u) = e =$ const.], then $p =$ const. Therefore, a Lagrangian systems of particles with constant rest mass density ρ_0 could exists moving on auto-parallel trajectories in a space with pressure p proportional to the kinetic energy of the particles. The contravariant vector field ξ is transported along u in a special way depending on the characteristics of u and the $(\overline{L}_n, g)$-space.

Special case: $\rho = \rho_0 :=$ const. $\neq 0$, $a_0 = 0 : p = p = p_0 + h_0 \cdot \rho_0 \cdot g(a, \xi)$.
The Euler-Lagrange equations for u and ξ will have the form

$$\nabla_u u = a = 0 \ , \tag{105}$$

$$\nabla_u \xi = (q - \delta u) \cdot \xi + (\xi)(g)(k_0) - \overline{N} \ \ . \tag{106}$$

If the Euler-Lagrange equations are fulfilled, then $p = p_0 =$ const. The auto-parallel equation appears in its canonical form. Therefore, a Lagrangian system of particles with constant rest mass density ρ_0 could exist moving on auto-parallel trajectories in a space with constant pressure p.

4.2.2 Auto-parallel equation as Euler-Lagrange's equation related to a frame of reference

If we use the basic arguments for introducing a generalized definition of a frame of reference (FR) [The set $FR \sim [u,\, T^{\perp u}(M),\, \nabla = \Gamma,\, \nabla_u]$ is called frame of reference [30] in a differentiable manifold M considered as a model of the space or of the space-time] we can also find a solution of the G-A problem by the use of the method of Lagrangians with covariant derivatives (MLCD).

Let us define a Lagrangian invariant in the form

$$\begin{aligned} L &= p_0 + h_0 \cdot g[\nabla_u(\rho \cdot u),\xi] = p_0 + h_0 \cdot g_{\overline{ij}} \cdot (\rho u^i)_{;k} \cdot u^k \cdot \xi^j \ , \\ p_0, \quad h_0 &= \text{const.}, \ \ \rho \in C^r(M) \ , \quad u,\ \xi \in T(M) \ . \end{aligned} \tag{107}$$

with the additional condition for the contravariant vector fields u and ξ: $g(u, \xi) = l = 0$. The corresponding action S could be written in the form

$$S = \int \sqrt{-d_g} \cdot (L + \lambda \cdot l) \cdot d^{(n)}x = \int (L + \lambda \cdot l) \cdot d\overline{\omega} \ , \qquad d_g = \det(g_{ij}) < 0 \ . \tag{108}$$

where λ is a Lagrangian multiplier. L is interpreted as the pressure p of a physical system, u is the velocity of the particles, ρ is their proper mass density, and ξ is a vector, orthogonal to u. By the use of the MLCD we obtain the covariant Euler-Lagrange equations for the vector fields u and ξ obeying the condition $l = 0$

$$\frac{\delta L}{\delta \xi^i} = 0 : \qquad u^i{}_{;j} \cdot u^j = [\frac{\lambda}{h_0} - u(\log \rho)] \cdot u^i \ , \tag{109}$$

$$\begin{aligned} \frac{\delta L}{\delta u^i} &= 0 : \quad \xi^i{}_{;j} \cdot u^j = (q - \delta u + \frac{\lambda}{h_0}) \cdot \xi^i + g_{\overline{kl}} \cdot u^k{}_{;j} \cdot g^{ji} \cdot \xi^l - \\ & \quad -[g^{ij} \cdot (g_{\overline{jk}})_{;m} \cdot u^m - g^i_{k;j} \cdot u^j] \cdot \xi^k \ . \end{aligned} \tag{110}$$

$$\frac{\delta L}{\delta \lambda} = 0 : g(u, \xi) = l = 0 \ . \tag{111}$$

In index-free form the equations for u and ξ would have the forms:

$$\nabla_u u = k \cdot u \ , \qquad k = \frac{\lambda}{h_0} - u(\log \rho) \ , \tag{112}$$

$$\nabla_u \xi = m \cdot \xi + K - \overline{N} \ , \qquad m = q - \delta u + \frac{\lambda}{h_0} \ , \tag{113}$$

$$q = q_j \cdot u^j \ , \quad q_j = T_{kj}{}^{k} - \frac{1}{2} \cdot g^{\overline{kl}} \cdot g_{kl;j} + g^l_k \cdot g^k_{l;j} \ , \tag{114}$$

$$\delta u = u^k{}_{;k} \ , \quad K = K^i \cdot \partial_i = (g_{\overline{kl}} \cdot u^k{}_{;j} \cdot g^{ji} \cdot \xi^l) \cdot \partial_i \ , \tag{115}$$

$$\overline{N} = \overline{N}^{\,i} \cdot \partial_i \ , \quad \overline{N}^{\,i} = [g^{ij} \cdot (g_{\overline{jk}})_{;m} \cdot u^m - g^i_{k;j} \cdot u^j] \cdot \xi^k \ . \tag{116}$$

The Euler-Lagrange's equation (112) is just the auto-parallel equation in a non-canonical form. For $\rho =$ const., it will have the form $\nabla_u u = \frac{\lambda}{h_0} \cdot u$. After changing the parameter of the curve to which u is a tangent vector field the auto-parallel equation could be found in its canonical form $\nabla_{\overline{u}} \overline{u} = 0$.

The Euler-Lagrange's equation for ξ (113) has in general a more complicated form than the parallel equation for ξ along u ($\nabla_u \xi = g \cdot \xi$). For different affine connections (and the corresponding models of space-time) this equation would have different solutions. Therefore, if we consider an auto-parallel equation as a result of a variational principle we should take into account the corresponding orthogonal to u sub space.

Remark 1 *The Lagrangian invariant L could be defined without the requirement ξ to be orthogonal to u. The covariant Euler-Lagrange's equations will be then found for u and a vector field $\xi \in T(M)$. For $\rho =$ const. the auto-parallel equation will have its canonical form. The orthogonality condition for u and ξ could be introduced after solving the Euler-Lagrange equations for u or ξ.*

5 Conclusion

In this paper we have considered the finding out of the auto-parallel equation in spaces with affine connections and metrics by the use of Lagrangian formalism related to the method of Lagrangians with covariant derivatives. If an appropriate Lagrangian density is taken into account, then the auto-parallel equation could be found as Euler-Lagrange's equation for an auto-parallel transported contravariant vector field. It was already shown that this type of equation could be interpreted as an equation for the motion of a free moving test particles in a space with affine connections (considered as a model of the space-time). The derivation of the auto-parallel equation on the basis of a Lagrangian formalism proves once more that this equation could be of use in field theories over more general spaces than the (pseudo) Riemannian spaces with or without torsion.

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